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# CYCLIC NORMAL SUBGROUPS OF FUNDAMENTAL GROUPS OF 3-MANIFOLDS

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Let $M$ be a compact, connected, orientable 3-manifold. If $M$ is a Seifert fibre space (other than $S^3$), then $\pi_1(M)$ has a non-trivial cyclic normal subgroup, namely that generated by the class of an ordinary fibre. If the orbit-surface (*Zerlegungsfläche*) of the fibring is orientable, then this subgroup is central. Conversely, Waldhausen has shown[11] that if $M$ is irreducible and sufficiently large (in the sense of [12]), and if $\pi_1(M)$ has a non-trivial centre, then $M$ has a Seifert fibring with orientable orbit-surface. By an argument based on Waldhausen's we prove the following (which was announced in [4]).

Theorem. *Let $M$ be a compact, connected, orientable, irreducible and sufficiently large 3-manifold. If $\pi_1(M)$ contains a (non-trivial) cyclic normal subgroup, then either*

(i) *$M$ is a Seifert fibre space or*

(ii) *there exists a closed surface $F$ in $M$ which separates $M$ into two twisted line bundles over a non-orientable surface $G$.*

In particular, if $\partial M \neq \emptyset$, then $M$ is a Seifert fibre space. We do not know whether there are any manifolds $M$ of type (ii), such that $\pi_1(M)$ has a non-trivial cyclic normal subgroup, which are not Seifert fibre spaces. H. Zieschang[13] claims a stronger theorem, using Smith Theory, which implies that the manifolds in case (ii) are in fact Seifert fibre spaces. However our proof is elementary in the sense that no deep results about Fuchsian groups are used.

Unless otherwise stated, from now on all 3-manifolds will be compact, connected, orientable and irreducible, and all surfaces compact, connected, orientable and properly embedded (i.e. if $F \subset M$, then $F \cap \partial M = \partial F$). If $F \subset M$, $U(F) \cong F \times I$ will denote a regular neighborhood of $F$ in $M$. For notational convenience, we shall sometimes not distinguish between a closed curve and the corresponding element of the appropriate fundamental group. A 3-manifold is *sufficiently large* if it contains an incompressible surface.

First we have the following purely algebraic result.

Lemma 1. *If $N$ is a cyclic normal subgroup of a proper free product with amalgamation $A *_C B$ such that $N \not\subset C$, then $C$ is of index 2 in $A$ and $B$.*

*Proof.* Choose coset representatives for $C$ in $A$ and $B$. Then every $x \in A *_C B$ has a unique expression as a product $cg_1 \dots g_r$, where $c \in C$, and the $g_i$ are non-trivial coset representatives of $C$ in alternate factors; $r = \lambda(x)$ is the *length* of $x$.

Let $z$ be a generator of $N$. Then for each $g \in A *_C B$, $g^{-1}zg = z^{n(g)}$ for some integer $n(g)$. Note that since $z^{n(g)n(g^{-1})} = z$, $g^{-1}zg$ is also a generator of $N$. Since $N \not\subset C$, $\lambda(z) \neq 0$. Also, if $\lambda(z) = 1$, then $\lambda(z^n) \leq 1$ for all $n$, whereas if $g$ belongs to the factor not containing $z$, and $g \notin C$, then $\lambda(g^{-1}zg) = 3$. Hence $\lambda(z) \geq 2$. Now let $z$ be chosen so that $\lambda(z)$ is minimal, so we have $z = cg_1 \dots g_r$, say, $r \geq 2$. Then $g_1$ and $g_r$ belong to different factors, for otherwise $g_r z g_r^{-1}$ would be a generator of $N$ of length $< r$. So assume without loss of generality that $g_1 \in A$ and $g_r \in B$. Let $g$ be any element of $B - C$. If $g_r g \notin C$, then $\lambda(g^{-1}zg) = r + 1$. But, since $g_1$ and $g_r$ belong to different factors, $\lambda(z^n) = |n|r$ for all $n$. Since $r \geq 2$, it follows that $g_r g \in C$. Hence $C$ is of index 2 in $B$. The same argument applied to $z^{-1}$ shows that $C$ is also of index 2 in $A$.

Lemma 2. *Let $M'$ be a Seifert fibre space with $\partial M' \neq \emptyset$. Then the subgroup $\{h\}$ generated by a*

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*fibre of $\partial M'$ is the unique maximal cyclic normal subgroup of $\pi_1(M')$, unless $M'$ is $S^1 \times D^2$, $S^1 \times S^1 \times I$, or the orientable $S^1$-bundle over the Möbius band.*

*Proof.* This essentially follows from [8, Satz 1]. The relevant exceptional cases are ($l = 0$ throughout):

$$\begin{array}{llll} p=0, & n\le 1, & m=1: S^1\times D^2; & \\ p=0, & n=0, & m=2: S^1\times S^1\times I; & \\ k=1, & n=0, & m=1 & \left.\begin{array}{l}\\ \end{array}\right\}\text{:orientable } S^1\text{-bundle over} \\ p=0, & n=2, & m=1, \quad \alpha_1=\alpha_2=2 & \phantom{\}}\text{Möbius band.} \end{array}$$

(The last case, which is omitted in [8], was kindly communicated to us by E. Vogt. It can be described as follows. Let $\alpha : S^1 \to S^1$ by reflection in some diameter. Let $I_0$ be a closed interval, and let $\beta : I_0 \to I_0$ be reflection about its mid-point. Let $A$ be the annulus $S^1 \times I_0$, and let $f : A \to A$ be $\alpha \times \beta$. Now consider $A \times I/f$, that is, $A \times I$ with $A \times 0$ and $A \times 1$ identified by the homeomorphism $f$. This is clearly homeomorphic to the orientable $S^1$-bundle over the Möbius band $I_0 \times I/\beta$. On the other hand, the $I$-fibres in $A \times I$ define a Seifert fibring of $A \times I/f$ with two exceptional fibres (coming from the two fixed points of $f$), which is easily seen to be $p = 0$, $n = 2$, $m = 1$, $\alpha_1 = \alpha_2 = 2$.)

LEMMA 3. *Let $M'$ be one of the exceptional manifolds listed in Lemma 2. If $k$ is a simple closed curve on $\partial M'$ such that some power of $k$ generates a non-trivial cyclic normal subgroup $N$ of $\pi_1(M')$, then there exists a Seifert fibring of $M'$ with $k$ as fibre.*

*Proof.* The lemma is clear if $M'$ is $S^1 \times D^2$ or $S^1 \times S^1 \times I$. So let $M'$ be the orientable $S^1$-bundle over the Möbius band. Let $t$, $h$ be generators of the fundamental group of the base and fibre respectively. Then $\pi_1(M') = \{t, h : t^{-1}ht = h^{-1}\}$. Suppose $t^m h^n \in N$. Then, in particular, $t^{-1}(t^m h^n)t = (t^m h^n)^\epsilon$, where $\epsilon = \pm 1$. If $\epsilon = +1$, then $h^{-n} = h^n$, which implies that $n = 0$. If $\epsilon = -1$, then mapping to the quotient group $\pi_1(M')/\{h\}$ shows that $m = 0$. Since $k$ is a simple closed curve on the torus $\partial M'$ whose fundamental group is generated by $h$ and $t^2$, we see that (up to orientation) $k = h$ or $k = t^2$. The two desired Seifert fibrings are those given in the proof of Lemma 2.

Let $M$ be a 3-manifold, with $F \subset M$ an incompressible surface. Inclusion induces a monomorphism $\pi_1(F) \to \pi_1(M)$ (we take some point in $F$ as basepoint for $F$ and $M$). Suppose that $F$ does not separate $M$.

LEMMA 4. *If $\pi_1(M)$ has a cyclic normal subgroup $N$ such that $N \not\subset \pi_1(F)$, then $M$ is a Seifert fibre space.*

*Proof.* Let $M' = Cl(M - U(F))$, so that there are two copies $F_1$, $F_2$ of $F$ in $\partial M'$, corresponding to $F \times 0$, $F \times 1$ in $F \times I \cong U(F)$. Let $p : \tilde{M} \to M$ be the infinite cyclic covering obtained by taking a countably infinite number of copies of $M'$ and identifying $F_2$ in the $i$th copy with $F_1$ in the $(i+1)$st, for all $i$, as in [11, p. 513]. Then $\pi_1(\tilde{M})$ is an infinite free product with amalgamation $\ldots *A_{-1}*_{B_{-1}}A_0*_{B_0}A_1*\ldots$, where each $A_i \cong \pi_1(M')$, each $B_i \cong \pi_1(F)$, and the inclusions $B_i \to A_i$, $B_i \to A_{i+1}$ correspond to the monomorphisms induced by the inclusions in $M'$ of $F_2$, $F_1$ respectively. With suitable choice of basepoint for $\tilde{M}$, the composition $\pi_1(F) \cong B_0 \to \pi_1(\tilde{M}) \xrightarrow{p_*} \pi_1(M)$ is the monomorphism induced by the inclusion of $F$ in $M$.

Let $t \in \pi_1(M)$ be the class of a simple closed curve in $M$ which pierces $F$ only at the basepoint. Then $\pi_1(M)$ is an extension of $\pi_1(\tilde{M})$ by the infinite cyclic group $\{t\}$. Let $z = t^m x$, $x \in \pi_1(\tilde{M})$, be an element of $N$. Then for $g \in \pi_1(M)$, we have $g^{-1}(t^m x)g = (t^m x)^{n(g)}$ for some integer $n(g)$.

If $m \neq 0$, then mapping the above equation to $\{t\}$ shows that $n(g) = 1$ for all $g \in \pi_1(M)$, and hence $z \in$ centre $(\pi_1(M))$. Waldhausen's arguments ([11], proof of (4.1) in case that centre $(\pi_1(M)) \not\subset \pi_1(F)$) then allow us to conclude that $M$ is a Seifert fibre space.

In fact, this is the only case that can occur. For suppose $N \subset \pi_1(\tilde{M})$. Write $\pi_1(\tilde{M}) = A^- *_{B_0} A^+$, where $A^- = \cdots *A_{-1}*_{B_{-1}}A_0$, and $A^+ = A_1 *_{B_1} A_2 * \ldots$. By Lemma 1, either (i) $N \subset B_0$, or (ii) $B_0$ has index 2 in $A^-$ and $A^+$, or (iii) $B_0 \to A^-$ or $A^+$ is onto. Case (i) does not occur since $N \not\subset \pi_1(F)$. In case (ii), $\pi_1(\tilde{M})$ is finitely generated, and it follows from Neuwirth's argument ([7], p. 31) that $B_0 \to A^-$ and $B_0 \to A^+$ are onto. In case (iii), it follows from [1] that again $B_0 \to A^-$ and $B_0 \to A^+$ are onto. But this implies that $N \subset \pi_1(\tilde{M}) = \pi_1(F)$, contrary to hypothesis.

Again let $F \subset M$ be a non-separating incompressible surface, and let $M' = Cl(M - U(F))$.

LEMMA 5. *If $\pi_1(M)$ has a non-trivial cyclic normal subgroup $N$ such that $N \subset \pi_1(F)$, and if $M'$ is a Seifert fibre space, then $M$ is a Seifert fibre space.*

*Proof.* Let $F_1$, $F_2$ be the two copies of $F$ in $\partial M'$, as in the proof of Lemma 4. We may regard $M$ as being obtained from $M'$ by identifying $F_1$ and $F_2$. Inclusions induce monomorphisms $\pi_1(F_i) \to \pi_1(M') \to \pi_1(M)$, $i = 1, 2$. Let $G_i$ be the component of $\partial M'$ containing $F_i$, $i = 1, 2$. Note that $F$ is a torus or annulus, $G_1$, $G_2$ are tori, and we may have $G_1 = G_2$. Let $k$ be a simple closed curve in $F$ such that $k^m$ generates $N$, for some $m$. For $i = 1, 2$, let $k_i$ be the copy of $k$ in $F_i$; then $\{k_i^m\}$ is a non-trivial cyclic normal subgroup of $\pi_1(M')$.

*Case (a).* $M'$ is not one of the exceptions listed in Lemma 2. Then, if $h_1$ is a fibre of $G_1$, for some $n$ we have $k_1^m \simeq h_1^n$ in $G_1$. Since $k_1$ and $h_1$ are simple closed curves on a torus, it follows (ignoring the orientation of $h_1$) that $k_1 \simeq h_1$, and hence $k_1$ is isotopic to $h_1$. We may thus assume that $k_1 = h_1$. Similarly, we may assume that $k_2$ is a fibre of $G_2$. It follows that we may deform the fibring of $M'$ near $F_1$ and $F_2$ so that the homeomorphism by which we identify $F_1$ and $F_2$ to get $M$ is fibre-preserving. Hence $M$ is a Seifert fibre space.

*Case (b).* $M'$ is one of the exceptions listed in Lemma 2. By Lemma 3 there is a Seifert fibring of $M'$ with $k_1$ as fibre. If $G_1 = G_2$, then (up to orientation) $k_1$ must be isotopic to $k_2$, and $M$ is a Seifert fibre space as above. If $G_1 \neq G_2$, then $M'$ is $S^1 \times S^1 \times I$. Let $t \in \pi_1(M)$ be as in the proof of Lemma 4. Then $t^{-1}k^m t = k^{\pm m}$. Hence $k_1^m \simeq k_2^{\pm m}$ (in $M'$), and therefore $k_1 \simeq k_2^{\pm 1}$. Hence $k_2$ is isotopic to a fibre of $G_2$, and again $M$ is a Seifert fibre space.

Finally, let $F \subset M$ be an incompressible surface which separates $M$, into $M_1$ and $M_2$, say. Then $\pi_1(M) = \pi_1(M_1) *_{\pi_1(F)} \pi_1(M_2)$. By an argument similar to the proof of Lemma 5 we have

LEMMA 6. *If $\pi_1(M)$ has a non-trivial cyclic normal subgroup $N$ such that $N \subset \pi_1(F)$, and if $M_1$ and $M_2$ are Seifert fibre spaces, then $M$ is a Seifert fibre space.*

*Proof of theorem. Case (a).* $\partial M \neq \emptyset$. By [11] or [2] there exists a sequence of 3-manifolds $M = M_0 \supset M_1 \supset \cdots \supset M_n = $ 3-ball, and incompressible non-separating surfaces $F_i \subset M_i$ such that $M_{i+1} = Cl(M_i - U(F_i))$, $0 \leq i < n$. Inclusions induce monomorphisms $\pi_1(F_i) \to \pi_1(M_i)$ and $\pi_1(M_{i+1}) \to \pi_1(M_i)$.

There are two possibilities:

(i) $N \subset \pi_1(F_i)$ for each $0 \leq i < n - 1$. Note that $M_{n-1}$ is a solid torus, and hence a Seifert fibre space. Then applying Lemma 5 $(n-1)$ times shows that $M$ is a Seifert fibre space.

(ii) There exists $0 \leq i < n - 1$ such that $N \not\subset \pi_1(F_i)$. Let $m$ be the least such $i$. Then $N \not\subset \pi_1(F_m)$, so $M_m$ is a Seifert fibre space by Lemma 4. Applying Lemma 5 $m$ times now shows that $M$ is a Seifert fibre space.

*Case (b).* $\partial M = \emptyset$. Let $F \subset M$ be an incompressible surface. There are two possibilities:

(i) $F$ does not separate $M$. If $N \not\subset \pi_1(F)$, then $M$ is a Seifert fibre space by Lemma 4. If $N \subset \pi_1(F)$, then $M' = Cl(M - U(F))$ is a Seifert fibre space by Case (a) above, and hence $M$ is a Seifert fibre space by Lemma 5.

(ii) $F$ separates $M$, into $M_1$ and $M_2$, say. Then $\pi_1(M) = \pi_1(M_1) *_{\pi_1(F)} \pi_1(M_2)$. By Lemma 1, either $N \subset \pi_1(F)$ or $\pi_1(F)$ has index $\leq 2$ in $\pi_1(M_i)$, $i = 1, 2$. In the first case, $M_1$ and $M_2$ are Seifert fibre spaces by Case (a), and hence $M$ is a Seifert fibre space by Lemma 6. In the second case, each $M_i$ is a twisted line bundle over a closed non-orientable surface $G$ by [3, Theorem 1].

The following Corollary gives a partial answer to Question 9 of [6].

COROLLARY. *Let $M$ be an irreducible, orientable, sufficiently large 3-manifold. If $M$ is covered by a compact Seifert fibre space $\tilde{M}$, then $M$ is either a Seifert fibre space or the union of two twisted line bundles over a closed non-orientable surface.*

*Proof.* By the following trick (suggested by H. Zieschang in a letter to us) we can assume that the covering $p: \tilde{M} \to M$ is regular:

Since $p_* \pi_1(\tilde{M})$ has finite index in $\pi_1(M)$, the normal subgroup $G = \cap\, g^{-1}\pi_1(\tilde{M})g$

($g \in \pi_1(M)$) has finite index in $\pi_1(M)$, and if $\hat{M}$ denotes the regular covering space of $M$ associated with $G$, we have finite coverings

$$\begin{tikzcd} & \hat{M} \arrow[dl] \arrow[dr] & \\ \tilde{M} \arrow[rr, "p"] & & M \end{tikzcd}$$

Since $\tilde{M}$ is a Seifert fibre space, so is $\hat{M}$.

Thus assume that $p : \tilde{M} \to M$ is a finite regular covering. If $\pi_1(\tilde{M})$ has a unique maximal cyclic normal subgroup $N$, then $N$ is characteristic in $\pi_1(\tilde{M})$ and hence normal in $\pi_1(M)$ and we can apply the theorem to $M$.

So we have to examine the cases that $\tilde{M}$ is a 'small' Seifert fibre space (in the sense of [9]).

(a) $\partial\tilde{M} \neq \emptyset$. Then $\tilde{M}$ is either

(i) $S^1 \times D^2$. In this case $\pi_1(\tilde{M})$ has a unique maximal cyclic normal subgroup.

(ii) $S^1 \times S^1 \times I$. Then $\partial M$ has one or two components. Let $T$ be the one which contains the basepoint for $\pi_1(M)$ and let $M'$ be the covering of $M$ associated to $\pi_1(T)$. We have subgroups $\pi_1(\tilde{M}) \to \pi_1(M') \to \pi_1(M)$ such that $\pi_1(\tilde{M})$ is normal and has finite index in $\pi_1(M)$. From [10, Lemma 1.6] and [3, Theorem 1] it follows that $M \approx S^1 \times S^1 \times I$ or $M$ is a twisted line bundle over a Kleinbottle. In both cases $M$ is a Seifert fibre space (by the theorem).

(iii) The orientable $S^1$-bundle over a Möbius band. In this case $\pi_1(\tilde{M})$ has a characteristic cyclic normal subgroup.

(b) $\partial\tilde{M} = \emptyset$. Note that $\tilde{M}$ is irreducible and sufficiently large. The 'small' closed Seifert fibre spaces are discussed in some detail by Orlik in [9, pp. 99–102, pp. 124–125]. Those which are orientable, irreducible, and sufficiently large are $S^1 \times S^1$-bundles over $S^1$, where the generator of the bundle group is the homeomorphism $S^1 \times S^1 \to S^1 \times S^1$ given by $\begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} -1 & -b \\ 0 & -1 \end{pmatrix}$ (the cases (iii), (iv), (ix) of Orlik). If $b \neq 0$ then it is not hard to show that $\pi_1(\tilde{M})$ has a unique maximal cyclic normal subgroup. If $b = 0$, then in the first case, $\tilde{M} \cong S^1 \times S^1 \times S^1$, and in the second case $\tilde{M}$ has $S^1 \times S^1 \times S^1$ as a two-fold covering.

Thus assume $\tilde{M} = S^1 \times S^1 \times S^1$. Note that any incompressible (connected) surface $\tilde{F}$ in $\tilde{M}$ is a nonseparating torus and $\tilde{M}$ cut along $\tilde{F}$ is homeomorphic to $\tilde{F} \times I$. Let $F$ be an incompressible surface in $M$. Cutting $\tilde{M}$ along all copies of $p^{-1}(F)$ gives copies of $S^1 \times S^1 \times I$ which cover $M$ cut along $F$. As in case (a) (ii) it follows that $M$ is either a union of two twisted line bundles over a Kleinbottle or $M$ is an $S^1 \times S^1$-bundle over $S^1$.

To show that $M$ is a Seifert fibre space in the latter case, it suffices to show that $M$ has finite bundle group. We have a short exact sequence

$$1 \to Z \times Z \to \pi_1(M) \xrightarrow{\beta} Z \to 1.$$

Let $t \in \pi_1(M)$ be an element such that $\beta(t)$ is a generator $\tau$ of $Z$. Then $t^{-1}xt = \phi(x)$ ($x \in Z \times Z$), where $\phi$ is the automorphism of $Z \times Z$ induced by the homeomorphism $S^1 \times S^1 \to S^1 \times S^1$ which generates the bundle group. Let $G \Delta \pi_1(M)$ be $\pi_1(\tilde{M}) \cong Z \times Z \times Z$. Since $G \not\subset Z \times Z$ it follows that $\beta(G) = \{\tau^k\}$ for some $k \geq 1$. Thus $G$ contains an element $t^k x$, $k \geq 1$, $x \in Z \times Z$. Since $G$ has finite index in $\pi_1(M)$, some powers $u^m$, $v^n$ of generators $u$, $v$ of $Z \times Z$ are in $G$. But since $G$ is abelian, $(t^k x)^{-1} u^m (t^k x) = u^m$, hence $(t^{-k} u t^k)^m = u^m$ and since $t^{-k} u t^k \in Z \times Z$, it follows that $t^{-k} u t^k = u$. Similarly, $t^{-k} v t^k = v$. Hence $\phi^k = 1$.

*Remark.* The coverings $S^1 \times S^1 \times S^1 \to M$ with finite cyclic group of covering transformations have been described in detail by J. Hempel [5].

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